Variational Principles for Water Waves

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Abstract

We describe the Hamiltonian structures, including the Poisson brackets and Hamiltonians, for free boundary problems for incompressible fluid flows with vorticity. The Hamiltonian structure is used to obtain variational principles for stationary gravity waves both for irrotational flows as well as flows with vorticity.

1 Introduction

In 1933 Friedrichs [9] proposed the functional

$$J(\psi) = \iiint_{0\leq\psi\leq 1} \left[(\nabla\psi)^2 + v^2(x,y)\right]\, d^2\mathbf{x},$$

where ψ is the stream function for an incompressible flow, as a variational method of obtaining solutions to free boundary value problems. Critical points of J are harmonic functions which satisfy the condition

$$(\nabla\psi)^2 = v^2$$

on the free boundary, given by $\psi = 1$. The free boundary condition relevant to theory of gravity waves, however, is the Bernoulli equation

$$\frac{(\nabla\varphi)^2}{2} + g\zeta = constant,$$

where φ is either the velocity potential for irrotational flow, or the stream function in the case of flows with vorticity. Thus some other variational principle is needed for the study of gravity waves.

Recently, a variational principle for gravity waves with vorticity was given by Constantin et. al. [7], using a direct, "hands on" approach. More generally, a variational principle for a stationary wave may be obtained for systems possessing a Hamiltonian structure by minimizing the Hamiltonian computed in a Galilean frame moving with the wave. We illustrate that approach in this study.

We begin with a brief review of Euler's equations of incompressible flows, and the associated free boundary value problems; in §3 we describe the Hamiltonian structure of these problems, for irrotational flows and flows with vorticity, as given by Lewis et.al. [15]. All the functions under consideration in this article, including the free boundaries, are assumed to be smooth.

2 Incompressible fluid flows

Let the velocity field of an incompressible fluid in a fixed region $\mathcal{D}$ be denoted by $\mathbf{v}$. The incompressibility of the fluid is expressed by the condition div $\mathbf{v} = 0$. We must have $\mathbf{v} \cdot \nu = 0$ on the boundary of $\mathcal{D}$, where ν is the outward unit normal at the boundary. Euler's equations of motion for the flow of an inviscid, incompressible fluid are

$$\frac{d\mathbf{x}}{dt} = \mathbf{v}, \qquad \rho \frac{d\mathbf{v}}{dt} = \rho(\mathbf{v}_t + (\mathbf{v} \cdot \nabla)\mathbf{v}) = -\nabla(\,p - \mathbf{g} \cdot \mathbf{x}), \tag{2.1}$$

$$\text{div } \mathbf{v} = 0,$$

where ρ is the density, p is the hydrodynamic pressure, and $\mathbf{g} \cdot \mathbf{x}$ is the gravitational potential. Henceforth we take $\rho = 1$.

Given a manifold $\mathcal{D} \in \mathbb{R}^3$ with smooth boundary, we denote by $\mathcal{L}^2(\mathcal{D})$ the Hilbert space of vector fields on $\mathcal{D}$ with the inner product

$$\langle \mathbf{v}, \mathbf{w} \rangle = \iiint_{\mathcal{D}} \mathbf{v} \cdot \mathbf{w} \, d^3\mathbf{x}.$$

We denote by $\mathcal{L}_\pi$ the closed subspace of $\mathcal{L}^2(\mathcal{D})$ generated by vector fields; of the form $\mathbf{w} = \nabla\, p$ for some function p with finite Dirichlet norm. The orthogonal complement $\mathcal{L}_\sigma = \mathcal{L}_\pi^\perp$ is the space of all vector fields $\mathbf{v}$ for which

$\langle \mathbf{v}, \nabla\, p\rangle = 0$ for all $p \in W^{1,2}(\mathcal{D})$. By applying the Gauss divergence theorem, we see that if $\mathbf{v} \in \mathcal{L}^2(\mathcal{D})$ and is smooth, say C^1, then div $\mathbf{v} = 0$ and $\mathbf{v} \cdot \nu = 0$ on $\Sigma = \partial\mathcal{D}$, where ν denotes the outward unit normal on Σ. The Hilbert space $\mathcal{L}_\sigma$ is the space of weakly divergence-free vector fields. We denote the orthogonal projections onto $\mathcal{L}_\sigma$ and $\mathcal{L}_\pi$ by P_σ and P_π respectively.

In many applications the fluid is not confined to a fixed region, but instead carries the region with it. In such cases, the region $\mathcal{D}$ occupied by the fluid must also be determined. Such problems are called free boundary problems and occupy a substantial part of the literature on incompressible flows.

Given an irrotational flow (curl $\mathbf{v} = 0$) on a simply connected domain, there is velocity potential φ for which $\mathbf{v} = \nabla\,\varphi$. The velocity potential is defined only up to an arbitrary function of time; the transformation $\varphi \mapsto \varphi + k(t)$ is called a *gauge transformation*, and will play a role in what follows.

The equation div $\mathbf{v} = 0$ implies that φ is harmonic. Substituting $\mathbf{v} = \nabla\,\varphi$ into the second equation in (2.1) we obtain

$$\nabla \left(\varphi_t + \frac{1}{2}(\nabla\,\varphi)^2 + gz + p \right) = 0,$$

hence

$$\varphi_t + \frac{1}{2}(\nabla\,\varphi)^2 + gz + p = k(t)$$

for some function of time, which can be eliminated by a gauge transformation of the velocity potential. We always choose the gauge to be such that

$$\varphi_t + \frac{1}{2}(\nabla\,\varphi)^2 + gz + p = 0$$

everywhere in the fluid.

An interface between the fluid and another medium, for example air, is called a *free surface*. If the pressure is constant in the air, then it is also constant at the surface of the fluid, and we may normalize the pressure to be zero at the free surface. Hence we obtain Bernoulli's equation

$$\varphi_t + \frac{1}{2}(\nabla\,\varphi)^2 + gz = 0,$$

where gz is the gravitational potential on the free surface.

The free surface is given in space-time by $\phi = 0$, where $\phi(x, y, z, t) = z - \zeta(x, y, t)$. The free surface moves with the fluid, hence the material derivative of ϕ vanishes, and

$$0 = \frac{d\phi}{dt} = \frac{d}{dt}(z - \zeta) = v^3 - \zeta_t - v^1\zeta_x - v^2\zeta_y;$$

hence

$$\zeta_t + v^1\zeta_x + v^2\zeta_y - v^3 = 0.$$

This is called the *kinematic* condition on the free surface.

This collection of equations for gravity waves on a free surface is known as Euler's equations for waves on the surface of an inviscid, incompressible fluid with irrotational flow in the region $\mathcal{D} = \{(x, y, z) : 0 \le z \le h + \zeta(x, y, t)\}$. They are

$$\Delta\varphi = 0 \qquad 0 \le y \le h + \zeta,$$

$$\zeta_t + \varphi_x\zeta_x + \varphi_y\zeta_y = \varphi_z \qquad \text{on } S; \tag{2.2}$$

$$\varphi_t + \frac{1}{2}|\nabla\varphi|^2 + g\,z = 0 \qquad \text{on } S; \tag{2.3}$$

$$\varphi_z = 0 \qquad \text{on } z = 0.$$

Here, φ is the velocity potential of the flow, and $\zeta(x, y, t)$ the displacement of the fluid surface from equilibrium. We have neglected surface tension. The second equation is known as the *kinematic equation*; the third equation is Bernoulli's equation. At rest, the fluid lies in the region $0 \le z \le h$; g is the acceleration due to gravity. The free surface is denoted by $S = \{(x, y, z) : z = h + \zeta(x, y, t)\}$.

The two physical constants in the theory are g and h. Let c denote a characteristic velocity (e.g. the velocity of a gravity wave); then h/c is a characteristic time. We introduce dimensionless variables

$$(x, y, z) = h(x', y', z'), \qquad t = ht'/c, \qquad \varphi = ch\varphi'.$$

The equation (2.3) now becomes

$$\varphi'_{t'} + \frac{1}{2}(\nabla'\,\varphi')^2 + \lambda\,\zeta = 0, \qquad \lambda = \frac{gh}{c^2}, \tag{2.4}$$

where λ is the inverse square of the *Froude* number. The other equations in Euler's system are unchanged under the rescaling. From now on we drop the primes and understand that we are working in non-dimensional variables.

The Euler equations are invariant under the one-parameter subgroup of Galilean boosts along the x axis, given by

$$(x', y', t') = (x - ct, y, t), \qquad \mathbf{v}'(x', y', t') = \mathbf{v}(x, y, t) - (c, 0).$$

The velocity potential, however, is determined only up to a function of time. Thus the Galilean boosts on the velocity potential are given by

$$\varphi'(x', y', t') = \varphi(x, y, t) - cx + q(t).$$

Under these Galilean boosts,

$$\frac{\partial \varphi'}{\partial t'} + \frac{1}{2}(\nabla' \varphi')^2 = \frac{\partial \varphi}{\partial t} + \frac{1}{2}(\nabla \varphi)^2 + q'(t) - \frac{1}{2}c^2. \tag{2.5}$$

The result follows by direct calculation, noting that

$$\frac{\partial}{\partial t'} = \frac{\partial}{\partial t} + c\frac{\partial}{\partial x}, \qquad \frac{\partial}{\partial x'} = \frac{\partial}{\partial x}.$$

Proposition 2.1. *Suppose the solutions of Euler's equations are stationary in a Galilean frame moving with speed c. Then* $\zeta_{t'} = \varphi'_{t'} = 0$*; and, choosing* $q(t) = c^2 t$*, the conditions on the free surface are (dropping the primes)*

$$\varphi_x \zeta_x = \varphi_y, \qquad \frac{\varphi_x^2 + \varphi_y^2}{2} + \lambda\, \zeta = \frac{c^2}{2}. \tag{2.6}$$

Proof. By (2.5) the Bernoulli equation in the moving frame is

$$\varphi'_{t'} + \frac{1}{2}(\nabla' \varphi')^2 + \lambda \zeta' = q'(t') - \frac{1}{2}c^2.$$

As $x \to \pm\infty$ $\zeta' \to 0$ while $(\nabla' \varphi')^2 \to c^2$. Moreover, $\varphi'_{t'} = 0$ by the assumption of stationarity. These conditions force the choice $q' = c^2$, and the result follows. The kinematic equation in the moving frame is immediate. □

Proposition 2.2. *Let* $\mathbf{v}$ *be a divergence-free vector field in a domain* $\mathcal{D}$*. There is a unique orthogonal decomposition, known as the Weyl-Hodge decomposition,*

$$\mathbf{v} = \mathbf{w} + \nabla \varphi, \tag{2.7}$$

$$\Delta\varphi = 0, \quad \varphi_\nu = \mathbf{v} \cdot \nu; \qquad \operatorname{div} \mathbf{w} = 0, \quad \mathbf{w} \cdot \nu = 0. \tag{2.8}$$

The proof is left to the reader.

3 Poisson structures

Let M be a C^∞ manifold of dimension n, and let $F, G \in C^\infty(M)$. A bilinear form $\{F, G\}$ is said to be a *Poisson bracket* if

- $\{F, G\} = -\{G, F\}$;
- $\{F, GH\} = \{F, G\}H + G\{F, H\}$.
- $\{\{F, G\}, H\} + \{\{G, H\}, F\} + \{\{H, F\}, G\} = 0$;

The second property implies that the Poisson bracket is a derivation in each of its entries. Hence any $H \in C^\infty(M)$ generates a vector field X_H, called a *Hamiltonian* vector field on M, defined by $X_H F = \{H, F\}$. The Hamiltonian vector field X_H generates a flow on M; if x^i are a set of local coordinates on M, then the time evolution of the x^i on that chart is given by the ordinary differential equations

$$\dot{x}^i = \{H, x^i\}.$$

Due to the fact that the bracket acts as a derivation on each of its entries, we may represent a Poisson bracket in the form

$$\{F, G\} = \sum_{j,k=1}^{n} W^{jk} \frac{\partial F}{\partial x^j} \frac{\partial G}{\partial x^k},$$

where $W^{jk}(x)$ is a skew-symmetric matrix.

If $\det W \neq 0$ then it is easily seen that n must be even. A classical theorem of Darboux states that in this case it is always possible to find a set of local coordinates, called *canonical* coordinates q^i, p^i, $(1 \leq i \leq n/2)$ in which the Poisson brackets take the form

$$\{F, G\} = \sum_{j=1}^{n} \frac{\partial F}{\partial p^j} \frac{\partial G}{\partial q^j} - \frac{\partial F}{\partial q^j} \frac{\partial G}{\partial p^j}.$$

A manifold with a Poisson bracket is called a *Poisson manifold*; if the brackets are non-degenerate, the manifold is called a *symplectic* manifold. On a symplectic manifold, the Hamiltonian flow takes the form

$$\dot{q}^i = \frac{\partial H}{\partial p^i}, \qquad \dot{p}^i = -\frac{\partial H}{\partial q^i}.$$

In this paper we shall restrict ourselves to the case in which M is a linear vector space with an inner product $\langle\ ,\ \rangle$; and we shall write the Poisson brackets in the form

$$\{F, G\} = \langle \nabla F, J_x \nabla G \rangle,$$

where J_x is a skew-symmetric linear transformation on M and ∇F is the gradient of the function F. The gradient is characterized as follows. Differentiating $F(x(t))$ along a curve $x(t)$ on M, we have

$$\frac{d}{dt}F = \langle \nabla F, \dot{x} \rangle.$$

If J_x is non-singular, then the Poisson brackets are non-degenerate and have locally a canonical system of coordinates. In many problems of physical interest, however, the Poisson brackets are degenerate, i.e. $\det J_x = 0$. For example, in the study of rigid motions about a fixed point in $\mathbb{R}^3$, the Poisson bracket is

$$\{F, G\} = \langle \nabla F, \mathbf{x} \times \nabla G \rangle. \tag{3.1}$$

The operator $J_{\mathbf{x}}$ is defined by $J_{\mathbf{x}}\mathbf{v} = \mathbf{x} \times \mathbf{v}$; hence $\ker(J_{\mathbf{x}}) = \mathbb{R}\mathbf{x}$.

The bracket (3.1) vanishes for all regular functions G whenever F is spherically symmetric. Such a function F is called a *Casimir*. It is invariant under any Hamiltonian flow generated by these brackets.

Any Poisson bracket on an odd-dimensional manifold must be degenerate and therefore have Casimirs. The bracket (3.1) is an example of a non-canonical Poisson bracket.

The formalism of Poisson brackets and Hamiltonian flows can be extended to infinite dimensions, for example, in the study of continuum mechanics, though a number of technical difficulties arise. In particular, Poisson structures play a useful role in the theory of the Euler equations for an incompressible fluid. Two important such brackets are the Poisson bracket introduced by Arnold [3, 4, 5] in his study of incompressible fluids on fixed domains, and the Poisson bracket implicit in Zakharov's fundamental discovery [18] of the Hamiltonian structure of the Euler equations of gravity waves.

3.1 Arnold's Poisson brackets

Arnold observed that Euler's equations for an incompressible fluid in a fixed domain $\mathcal{D}$ are directly analogous to his equations for rigid body motion, and that they have a Hamiltonian structure with the Hamiltonian and Poisson brackets given respectively by

$$H = \iiint\limits_{\mathcal{D}} \frac{1}{2}\mathbf{v}\cdot\mathbf{v}\, d^3\mathbf{x}, \tag{3.2}$$

and

$$\{F, G\} = \iiint\limits_{\mathcal{D}} \frac{\delta F}{\delta \mathbf{v}} \cdot \left(\text{curl } \mathbf{v} \times \frac{\delta G}{\delta \mathbf{v}}\right) d^3\mathbf{x}. \tag{3.3}$$

Here, F and G are functionals on $\mathcal{L}_\sigma$ with gradients in $\mathcal{L}_\sigma$. The gradient of F is $\delta F/\delta\mathbf{v}$, the Euler-Lagrange derivative of F with respect to $\mathbf{v}$. For example, $\frac{\delta H}{\delta \mathbf{v}} = \mathbf{v}$. The operator $J_{\mathbf{v}}$ in this case is

$$J_{\mathbf{v}}\mathbf{w} = P_\sigma(\text{curl } \mathbf{v} \times \mathbf{w}).$$

Let us show that (2.1) are the Hamiltonian equations generated by (3.2) and (3.3). We have

$$\dot{F} = \iiint\limits_{\mathcal{D}} \frac{\delta F}{\delta \mathbf{v}} \cdot \mathbf{v}_t\, d^3\mathbf{x},$$

$$\{H, F\} = \iiint\limits_{\mathcal{D}} \frac{\delta H}{\delta \mathbf{v}} \cdot (\text{curl } \mathbf{v} \times \frac{\delta F}{\delta \mathbf{v}})\, d^3\,\mathbf{x} = \iiint\limits_{\mathcal{D}} \frac{\delta F}{\delta \mathbf{v}} \cdot (\mathbf{v} \times \text{curl } \mathbf{v})\, d^3\,\mathbf{x}.$$

The Hamiltonian flow $\dot{F} = \{H, F\}$ implies that

$$\iiint\limits_{\mathcal{D}} \frac{\delta F}{\delta \mathbf{v}} \cdot (\mathbf{v}_t + (\text{curl } \mathbf{v}) \times \mathbf{v})\, d^3\mathbf{x} = 0 \tag{3.4}$$

for all admissible F on $\mathcal{L}_\sigma$.

All linear functionals of the form $F_{\mathbf{w}}(\mathbf{v}) = \langle \mathbf{w}, \mathbf{v}\rangle$ are admissible, and the gradient of $F_{\mathbf{w}}$ is the vector $\mathbf{w}$. Therefore $\mathbf{v}_t + (\text{curl } \mathbf{v}) \times \mathbf{v}$ belongs to $\mathcal{L}_\sigma^\perp = \mathcal{L}_\pi$. Hence it is a gradient, and

$$\mathbf{v}_t + (\text{curl } \mathbf{v}) \times \mathbf{v} = \nabla f$$

for some function f. The Euler momentum equations (2.1) follow from this and the vector identity

$$(\text{curl } \mathbf{v}) \times \mathbf{v} = (\mathbf{v}\cdot\nabla)\mathbf{v} - \frac{1}{2}\nabla\,|\mathbf{v}|^2, \tag{3.5}$$

if we let $f = -p + \frac{1}{2}\nabla|\mathbf{v}|^2$.

Just as in the case of rigid motion, the Arnold bracket is degenerate. This degeneracy is related to the action of the (formal) group of volume preserving diffeomorphisms acting on $\mathcal{D}$. Arnold's Poisson bracket is an example of a *Lie-Poisson* bracket.

3.2 Zakharov's Poisson brackets

In 1968, Zakharov made a striking observation: Euler's equations for *irrotational gravity waves* have a canonical Hamiltonian structure. The Hamiltonian (in non-dimensional variables) is

$$H = \frac{1}{2}\iiint\limits_{\mathcal{D}} (\nabla\varphi)^2\, d^3\mathbf{x} + \frac{1}{2}\lambda \iint\limits_{\mathbb{R}^2} \zeta^2(x,y,t)\, d^2\mathbf{x}.$$

The Poisson brackets implicit in Zakharov's observation are the canonical brackets

$$\{F,G\} = \iint\limits_{\mathbb{R}^2} \left(\frac{\delta F}{\delta\varphi}\frac{\delta G}{\delta\zeta} - \frac{\delta F}{\delta\zeta}\frac{\delta G}{\delta\varphi}\right) d^2\mathbf{x};$$

the Hamiltonian flow is then the canonical flow

$$\zeta_t = \frac{\delta H}{\delta\varphi}, \qquad \varphi_t = -\frac{\delta H}{\delta\zeta}.$$

The Hamiltonian H is regarded as a functional of $(\widetilde{\varphi}, \zeta)$ where $\zeta = \zeta(x,y,t)$ is the height of the free surface, and $\widetilde{\varphi} = \varphi|_S$ is the trace of the harmonic function φ on the free surface, with $\varphi_\nu = 0$ on the bottom. The evolution takes place in the space of harmonic functions on $\mathcal{D}$.

Zakharov's result is verified by calculating the gradients of H with respect to ζ and φ. Now

$$\frac{d}{d\varepsilon}H(\varphi,\zeta_\varepsilon)\Big|_{\varepsilon=0} = \iint\limits_{\mathbb{R}^2} \left[\frac{1}{2}(\nabla\widetilde{\varphi})^2 + \lambda\zeta\right]\delta\zeta\, d^2\mathbf{x},$$

where $\nabla\widetilde{\varphi}$ denotes $\nabla\varphi\big|_{\mathbb{R}^2}$. By identification,

$$\frac{\delta H}{\delta\zeta} = \frac{1}{2}(\nabla\widetilde{\varphi})^2 + \lambda\zeta.$$

Similarly,

$$\frac{d}{d\varepsilon}H(\widetilde{\varphi}_\varepsilon,\zeta)\Big|_{\varepsilon=0} = \iiint\limits_{\mathcal{D}} \nabla\,\varphi\cdot\nabla\,\delta\varphi\,d^3\mathbf{x}$$

$$= -\iiint\limits_{\mathcal{D}} \delta\varphi\Delta\varphi\,d^3\mathbf{x} + \iint\limits_{\Sigma} \delta\varphi\frac{\partial\varphi}{\partial\nu}dS = \iint\limits_{\Sigma} \delta\varphi\frac{\partial\varphi}{\partial\nu}dS,$$

since φ is harmonic in $\mathcal{D}$ and $\varphi_\nu = 0$ on the bottom.

On the free surface

$$\widetilde{\varphi}_\nu dS = \nabla\,\widetilde{\varphi}\cdot\frac{(-\zeta_x,-\zeta_y,1)}{\sqrt{1+\zeta_x^2+\zeta_y^2}}\sqrt{1+\zeta_x^2+\zeta_y^2}d^2\mathbf{x};$$

so

$$\frac{\delta H}{\delta\widetilde{\varphi}} = \widetilde{\varphi}_z - \widetilde{\varphi}_x\zeta_x - \widetilde{\varphi}_y\zeta_y.$$

The free boundary equations (2.2) and (2.3) are thus precisely the Hamiltonian equations for this system.

Remark. The effects of surface tension can be obtained by simply adding the boundary integral

$$\sigma\iint\limits_{\Sigma} dS$$

to the Hamiltonian, where σ is the coefficient of surface tension, and dS is the element of surface area on the free surface S. The inclusion of surface tension leads to an additional term in the Bernoulli equation; when the free surface is a graph $z = \zeta(x,y,t)$, it is

$$\varphi_t + \frac{1}{2}|\nabla\varphi|^2 + g\,z = \sigma\ \mathrm{div}\,\frac{\nabla\,\zeta}{\sqrt{1+(\nabla\,\zeta)^2}}, \qquad \nabla\,\zeta = (\zeta_x,\zeta_y).$$

The potential energy can also be written as the integral of the gravitational potential over the fluid domain, so that the Hamiltonian for gravity waves including the effects of surface tension is

$$H = \iiint\limits_{\mathcal{D}} \left[\frac{(\nabla\,\varphi)^2}{2} + \lambda\,U_+(\mathbf{x})\right]\,d^3\mathbf{x} + \sigma\iint\limits_{S} dS, \tag{3.6}$$

where $U_+(\mathbf{x})$ is the gravitational potential, truncated in such a way that the integral over the unbounded domain $\mathcal{D}$ converges. When the fluid is a horizontal layer and the gravity field is constant in the negative z direction, we take $U_+ = (z-1)_+$, where z_+ denotes the function given by z when $z > 0$ and by 0 when $z < 0$. The factor g has been absorbed into the pure parameter λ.

4 Free boundary flows with vorticity.

Free boundary value flows with vorticity, with both gravitational forces and surface tension included, are generated by the Hamiltonian

$$H = \iiint\limits_{\mathcal{D}} \mathcal{E}\, d^3\mathbf{x} + \sigma \iint\limits_{\Sigma} dS, \qquad \mathcal{E} = \frac{\mathbf{v}\cdot\mathbf{v}}{2} + \lambda\, U_+(\mathbf{x}) \tag{4.1}$$

The corresponding Poisson brackets are [15]

$$\{F, G\} = \iiint\limits_{\mathcal{D}} \frac{\delta F}{\delta \mathbf{v}} \cdot \left(\operatorname{curl} \mathbf{v} \times \frac{\delta G}{\delta \mathbf{v}}\right) d^3\mathbf{x} + \iint\limits_{\Sigma} \left(\frac{\delta F}{\delta \varphi}\frac{\delta G}{\delta \Sigma} - \frac{\delta F}{\delta \Sigma}\frac{\delta G}{\delta \varphi}\right) dS, \tag{4.2}$$

where Σ is the free boundary and dS is the element of surface area on Σ.

Admissible functionals are regarded as functions of $\mathbf{v}$ and Σ, the free boundary of $\mathcal{D}$, and their gradients are defined implicitly by the relation

$$\frac{d}{d\varepsilon} F(\mathbf{v}_\varepsilon, \Sigma_\varepsilon)\Big|_{\varepsilon=0} = \iiint\limits_{\mathcal{D}} \frac{\delta F}{\delta \mathbf{v}} \cdot \delta \mathbf{v}\, d^3\mathbf{x} + \iint\limits_{\Sigma} \frac{\delta F}{\delta \Sigma} \delta\Sigma\, dS.$$

Variations with respect to the free surface are restricted to normal variations, in a sense explained below. Admissible functionals F are those for which $\delta F/\delta \mathbf{v}$ is a divergence free vector field. We require that $\iint_{\mathcal{D}} \delta\Sigma dS = 0$, reflecting the fact that only volume preserving variations are allowed. This means that the gradient of a functional with respect to Σ is determined only up to a constant.

Let $\mathcal{L}_d(\mathcal{D})$ be the space of divergence free L^2 vector fields on $\mathcal{D}$. Let P_1 and P_2 be the orthogonal projections defined by $P_1\mathbf{v} = \mathbf{w}$ and $P_2\mathbf{v} = \nabla\varphi$ in the Weyl-Hodge decomposition.

Lemma 4.1. *Let*

$$F(\mathbf{v}, \Sigma) = \iiint\limits_{\mathcal{D}} \mathcal{F}(\mathbf{v}, \mathbf{x}) d^3\mathbf{x} + \sigma \iint\limits_{\Sigma} dS$$

be an admissible functional. Then

$$\frac{\delta F}{\delta \mathbf{w}} = P_1 \frac{\delta F}{\delta \mathbf{v}} \in \mathcal{L}^2(\mathcal{D}, d^3\mathbf{x}).$$

The gradients with respect to φ and Σ lie in $\mathcal{L}^2(\Sigma, dS)$ and are given by

$$\frac{\delta F}{\delta \varphi} = \frac{\delta F}{\delta \mathbf{v}}\Big|_{\Sigma} \cdot \nu, \qquad \frac{\delta F}{\delta \Sigma} = \mathcal{F}(\mathbf{v}, \mathbf{x}) + \sigma\kappa\Big|_{\Sigma} \quad \textit{mod constant},$$

where κ is the mean curvature function on Σ.

Proof. Applying the Weyl-Hodge decomposition to both $\delta\mathbf{v}$ and $\delta F/\delta\mathbf{v}$ we obtain

$$\left\langle \frac{\delta F}{\delta \mathbf{v}}, \delta\mathbf{v} \right\rangle = \iiint\limits_{\mathcal{D}} \frac{\delta F}{\delta \mathbf{v}} \cdot \delta\mathbf{v}\, d^3\mathbf{x} = \iiint\limits_{\mathcal{D}} P_1 \frac{\delta F}{\delta \mathbf{v}} \cdot \delta\mathbf{w} + P_2 \frac{\delta F}{\delta \mathbf{v}} \cdot \delta\nabla\varphi\, d^3\mathbf{x}.$$

By the uniqueness of the Weyl-Hodge decomposition, we may conclude

$$\frac{\delta F}{\delta \mathbf{w}} = P_1 \frac{\delta F}{\delta \mathbf{v}}, \qquad \frac{\delta F}{\delta \varphi} = P_2 \frac{\delta F}{\delta \mathbf{v}}.$$

Since $\delta F/\delta\mathbf{v}$ is divergence free, we have, by the divergence theorem,

$$\iiint\limits_{\mathcal{D}} P_2 \frac{\delta F}{\delta \mathbf{v}} \cdot \delta\nabla\varphi\, d^3\mathbf{x} = \iiint\limits_{\mathcal{D}} \frac{\delta F}{\delta \mathbf{v}} \cdot \nabla\delta\varphi\, d^3\mathbf{x} = \iint\limits_{\partial\mathcal{D}} \frac{\delta F}{\delta \mathbf{v}} \cdot \nu\, \delta\varphi\, dS,$$

and the second relation follows.

Let Σ_ε be a one parameter family of surfaces parameterized by a vector valued map

$$\mathbf{X}(u, v, \varepsilon) = \mathbf{X}_0(u, v) + \varepsilon\, \delta\Sigma\, \mathbf{N}(u, v),$$

where $\mathbf{N}$ is the normal vector field to Σ. For ε sufficiently small, the symmetric difference $\mathcal{D}_\varepsilon \Delta \mathcal{D}$ of the domains bounded respectively by Σ_ε and Σ is contained in a tubular neighborhood of Σ. In this neighborhood, the volume element of the 3-space can be written as $d^3\mathbf{x} = dr dS$ where dS is the area element on Σ and dr corresponds to the normal coordinate in the tubular neighborhood. We get

$$\delta \iiint\limits_{\mathcal{D}} \mathcal{F}(\mathbf{v}, \mathbf{x})\, d^3\mathbf{x} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \iiint\limits_{\mathcal{D}_\varepsilon \Delta \mathcal{D}} \mathcal{F}(\mathbf{v}, \mathbf{x})\, d^3\mathbf{x}$$

$$= \iint\limits_{\Sigma} \left(\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^{\varepsilon \delta \Sigma} \mathcal{F}(\mathbf{v}, \mathbf{x})\, dr \right) dS = \iint\limits_{\Sigma} \mathcal{F}(\mathbf{v}, \mathbf{x}) \delta\Sigma\, dS.$$

On the other hand, by classical differential geometry,

$$\delta \iint\limits_{\Sigma} dS = \iint\limits_{\Sigma} \kappa \delta \Sigma\, dS,$$

where κ is the mean curvature function on Σ. This completes the proof of Lemma 4.1. $\square$

Let us derive the equations of motion from the Hamiltonian structure. We have

$$\frac{\delta H}{\delta \mathbf{v}} = \mathbf{v}, \qquad \frac{\delta H}{\delta \varphi} = \mathbf{v} \cdot \nu, \qquad \frac{\delta H}{\delta \Sigma} = \mathcal{E}\Big|_{\Sigma} + \sigma\kappa.$$

¿From $\dot{F} = \{H, F\}$, we get

$$\iiint\limits_{\mathcal{D}} \frac{\delta F}{\delta \mathbf{v}} \cdot \mathbf{v}_t\, d^3\mathbf{x} + \iint\limits_{\Sigma} \frac{\delta F}{\delta \Sigma} \Sigma_t\, dS =$$

$$\iiint\limits_{\mathcal{D}} (\mathbf{v} \times (\mathrm{curl}\ \mathbf{v})) \cdot \frac{\delta F}{\delta \mathbf{v}}\, d^3\mathbf{x} + \iint\limits_{\Sigma} \left(\mathbf{v} \cdot \nu \frac{\delta F}{\delta \Sigma} - (\mathcal{E} + \sigma\kappa) \frac{\delta F}{\delta \varphi} \right) dS. \quad (4.3)$$

Since

$$\iint\limits_{\Sigma} \mathcal{E} \frac{\delta F}{\delta \varphi}\, dS = \iint\limits_{\Sigma} \mathcal{E} \frac{\delta F}{\delta \mathbf{v}} \cdot \nu\, dS = \iiint\limits_{\mathcal{D}} \nabla \mathcal{E} \cdot \frac{\delta F}{\delta \mathbf{v}}\, d^3\mathbf{x}$$

and $\delta F/\delta \mathbf{v}$ is divergence free, we get from (4.3), using functionals for which $\delta F/\delta\varphi = 0$,

$$\mathbf{v}_t + (\text{curl } \mathbf{v}) \times \mathbf{v} = \nabla(-p + \mathcal{E}), \qquad \Sigma_t = \mathbf{v} \cdot \nu\Big|_{\Sigma}.$$

The boundary condition on the bottom is $\mathbf{v} \cdot \nu = 0$, where ν is the outward normal.

The first equation, together with (3.5) imply that $\Delta p = -\operatorname{div}(\mathbf{v} \cdot \nabla)\mathbf{v}$. Substituting the two equations above into (4.3), we obtain

$$\iiint\limits_{\mathcal{D}} \nabla\, p \cdot \frac{\delta F}{\delta \mathbf{v}}\, d^3\mathbf{x} - \iint\limits_{\Sigma} \sigma\kappa \frac{\delta F}{\delta\varphi}\, dS = 0$$

for all admissible functionals F. Applying the divergence theorem to the integral over $\mathcal{D}$ we obtain

$$\iint\limits_{\Sigma} (p - \sigma\kappa) \frac{\delta F}{\delta\varphi}\, dS = 0,$$

for all admissible functionals F. But

$$\iint\limits_{\Sigma} \frac{\delta F}{\delta\varphi}\, dS = \iint\limits_{\Sigma} \frac{\delta F}{\delta \mathbf{v}} \cdot \nu\, dS = \iiint\limits_{\mathcal{D}} \operatorname{div} \frac{\delta F}{\delta \mathbf{v}}\, d^3\mathbf{x} = 0;$$

and therefore

$$p\Big|_{\Sigma} = \sigma\kappa + constant. \tag{4.4}$$

Thus the Hamiltonian approach yields the dynamic conditions on the free boundary in the case of surface tension. [7, 15, 12].

Remark. In the general theory one considers normal variations of the free surface, whereas in the theory of gravity waves on a free surface over a horizontal bottom, it is customary to use the height of the free surface, ζ. More generally, if the surface Σ is a graph over a fixed manifold $\mathcal{M}$, we may represent Σ by a "height" function ζ defined on $\mathcal{M}$. In that case we refer to $\delta\zeta$ as the "vertical" variation and $\delta\Sigma$ as the "normal" variation.

Proposition 4.2. *Let $\delta\Sigma$ and $\delta\zeta$ denote the normal and vertical variations of a surface Σ in the case when Σ is a graph over a fixed manifold. Let Σ be given in local coordinates by $\phi = 0$, where $\phi = z - \zeta$. Then $\delta\zeta = |\nabla\phi|\, \delta\Sigma$.*

Proof. Let $\mathbf{X} : U \mapsto \mathbb{R}^3$ be a local embedding of Σ in $\mathbb{R}^3$; and let $\mathbf{X}_\varepsilon$ be a one parameter family of embeddings, with $\mathbf{X}_0 = \mathbf{X}$. Then

$$\delta\Sigma = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} (\mathbf{X}_\varepsilon(u,v)) \cdot \nu.$$

Let Σ be defined by $\phi = 0$, $\phi = z - \zeta$. Then $\mathbf{X}_\varepsilon = (u, v, \zeta_\varepsilon(u,v)$; and

$$\delta\Sigma = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \begin{pmatrix} u \\ v \\ \zeta_\varepsilon(u,v) \end{pmatrix} \cdot \nu = \begin{pmatrix} 0 \\ 0 \\ \delta\zeta \end{pmatrix} \cdot \frac{\nabla\phi}{|\nabla\phi|} = \frac{\delta\zeta}{|\nabla\phi|}.$$

□

5 Variational principles for traveling waves

The Hamiltonian structure of the equations for gravity waves can be used to obtain variational principles for traveling waves – waves of constant speed and shape. Such a wave is a stationary solution of the Hamiltonian system in a Galilean frame moving with the wave; thus the wave is a critical point for the Hamiltonian, computed in such a reference frame. We apply the method here to the general case of gravity waves on a horizontal surface. The variational principle for irrotational flows given below appears to be new.

A variational approach, if successful, would permit a global treatment of the existence of traveling waves by the direct methods of the calculus of variation; but so far, the existence of traveling waves for potential flows of low amplitude have been proved by perturbation methods. The first existence theorems were given independently for periodic wave trains by Levi-Civita [14] and Struik [17] in the case of finite depth. The existence of the solitary wave, which is a more difficult problem, was first proved by Friedrichs and Hyers [10], since the bifurcation problem in this case is a singular perturbation problem (see the discussion by Sattinger [16]). These authors used conformal mapping techniques. A dynamical systems approach to the existence of traveling waves has been developed by Kirchgässner [13]; Amick and Toland [2] have shown that periodic wave trains tend to a solitary wave in the limit as the period tends to infinity.

In the direct method, one first uses compactness properties of the functional to obtain a minimum from a minimizing sequence. In general, this guarantees only a weak solution of the associated Euler-Lagrange equations.

In many cases, these are elliptic equations, and it is possible to prove sufficient regularity of the weak solution to show that in fact it is a classical solution to the problem. (See Alt and Caffarelli [1] for functionals of Friedrichs' type.) For the present, we simply indicate the method for the problems discussed here in the theorems below.

Theorem 5.1. *Euler's equations for gravity waves are the Euler-Lagrange equations for the functional*

$$\mathcal{H}(\varphi,\zeta) = \iint\limits_{\mathcal{D}_\zeta} \left[\frac{1}{2}\left[(\nabla\varphi)^2 - 1\right] + \lambda(y-1)_+\right] d^2\mathbf{x}\,, \tag{5.1}$$

where

$$y_+ = \begin{cases} 0, & y \le 0; \\ y, & y \ge 0. \end{cases}$$

$$\mathcal{D}_\zeta = \{(x,y) \,:\, -\infty < x < \infty,\ 0 \le y \le 1 + \zeta(x)\}\,;$$

and the minimum is taken over all functions φ for which

$$\iint\limits_{\mathcal{D}} \left[(\varphi_x - 1)^2 + \varphi_y^2\right] d^2\mathbf{x} < +\infty.$$

If (φ,ζ) is a local minimum of $\mathcal{H}$, then φ is harmonic on the interior of $\mathcal{D}_\zeta$; if ζ is C^1 and $\varphi \in H^2(\mathcal{D})$, then the kinematic and Bernoulli equations hold on the free surface.

Remark. The Hamiltonian (5.1) is the renormalization of the Hamiltonian in the moving frame. By carrying out the integration in y we obtain

$$\iint\limits_{\mathcal{D}} (y-1)_+\, d^2\mathbf{x} = \frac{1}{2}\int_{-\infty}^{\infty} \zeta^2\, dx;$$

thus $\mathcal{H}$ can also be written

$$\mathcal{H}(\varphi,\zeta) = \iint\limits_{\mathcal{D}} \frac{1}{2}\left[(\nabla\varphi)^2 - 1\right] d^2\mathbf{x} + \frac{\lambda}{2}\int_{-\infty}^{\infty} \zeta^2(x)\, dx.$$

If $\varphi \in H^2(\mathcal{D})$ and $\zeta \in C^1$, then $\nabla\varphi$ has an L^2 trace on the boundary $y = \zeta$, and Stokes theorem applies.

Proof. Let (φ,ζ) be a minimizer of $\mathcal{H}$ and suppose that ζ is C^1 and $\varphi \in H^2(\mathcal{D})$. Let $(\varphi_\varepsilon,\zeta_\varepsilon)$ be a one parameter family of admissible functions and denote the corresponding domains by $\mathcal{D}_\varepsilon$. By the calculations in §3.2 we have

$$\begin{aligned}\delta\mathcal{H}(\delta\varphi,\delta\zeta) &= \frac{\partial H(\varphi_\varepsilon,\zeta_\varepsilon)}{\partial\varepsilon}\Big|_{\varepsilon=0}\\ &= -\iint\limits_{\mathcal{D}} \Delta\varphi\delta\varphi\, d^2\mathbf{x} + \oint\limits_{\partial\mathcal{D}} \varphi_\nu\delta\varphi\, ds + \int_{-\infty}^{+\infty}\left[\frac{1}{2}(\nabla\varphi)^2 - \frac{1}{2} + \lambda\zeta\right]\delta\zeta\, dx\\ &= 0,\end{aligned}$$

for all admissible $\delta\varphi$, $\delta\zeta$.

Since the bottom is fixed, $\varphi_\nu = 0$ on $y=0$. We first restrict ourselves to variations for which $\delta\zeta = \delta\varphi\big|_S = 0$. Then the double integral must vanish for a set of variations $\delta\varphi$ which are dense in $L^2(\mathcal{D})$; it follows that φ is harmonic in the interior of $\mathcal{D}$. As before, $\varphi_\nu ds = \nabla\varphi\cdot(-\zeta_x,1)dx = (\varphi_y - \zeta_x\varphi_x)dx$; and so

$$\delta\mathcal{H} = \int_{-\infty}^{+\infty}\left[\left(\frac{(\nabla\varphi)^2-1}{2} + \lambda\zeta\right)\delta\zeta + (\varphi_y - \varphi_x\zeta_x)\,\delta\varphi\right]dx.$$

Setting first $\delta\zeta = 0$ and letting $\delta\varphi$ vary on Σ, we obtain the kinematic equation on the free surface. Therefore the second term always vanishes. Now allowing $\delta\zeta$ to vary, we see that Bernoulli's equation holds on Σ. □

6 A variational problem with constraint

Whereas Friedrich's paper shows that Bernoulli's equation is not obtained when the functional J is minimized with respect to the stream function, Constantin et. al. showed in [7] that traveling gravity waves in the rotational case are obtained as extremals of a variational problem for the stream function with constraints. The existence of traveling water waves with vorticity was established in [8] for the periodic case. In a recent PhD thesis at Brown University, V. Hur [11] has constructed solitary waves with non-zero vorticity. Some of their qualitative properties were investigated in [6].

In the irrotational case we have

Theorem 6.1. *Define the set of admissible functions $\mathcal{K} = \{\psi, \zeta\}$ with the following properties*

$$i) \qquad \int_{-\infty}^{+\infty} \zeta(x)dx = m; \ \int_{-\infty}^{+\infty} \zeta^2 \, dx < \infty;$$

$$ii) \qquad \psi(x,0) = 0; \ \psi(x, 1+\zeta(x)) = 1,$$

$$iii) \qquad \iint\limits_{\mathcal{D}} \left[\psi_x^2 + (\psi_y - 1)^2\right] d^2\mathbf{x} < +\infty.$$

Consider the variational problem

$$\lambda = \inf_{\mathcal{K}} \frac{\iint_{\mathcal{D}} \left[(\nabla\,\psi)^2 - 1\right] d^2\mathbf{x}}{\int_{-\infty}^{+\infty} \zeta^2 \, dx}.$$

Let (ψ, ζ) be a minimizer in $\mathcal{K}$ of the above variational principle. Then ψ is harmonic in the interior of $\mathcal{D}$. If ζ is C^1, and $\psi \in H^2(\mathcal{D})$, then the Bernoulli equation is satisfied on the free surface $\psi = 1$. Hence minima of the above variational problem provide an irrotational flow for the gravity wave problem.

Proof. Let (ψ, ζ) be a minimizer, and let $\psi_\varepsilon, \zeta_\varepsilon$ be a family of admissible functions with $\psi_0 = \psi$ and $\zeta_0 = \zeta$. Then $J(\varepsilon) \geq 0$ and $J(0) = 0$, where

$$J(\varepsilon) = \iint\limits_{\mathcal{D}_\varepsilon} \left[(\nabla\,\psi_\varepsilon)^2 - 1\right] \, d^2\mathbf{x} - \lambda \int_{-\infty}^{+\infty} \zeta_\varepsilon^2 \, dx.$$

Then $\delta J(\delta\psi, \delta\zeta) = 0$ for all admissible variations, where

$$\begin{aligned} \delta J &= \iint\limits_{\mathcal{D}} 2\,\nabla\,\psi \cdot \nabla\,\delta\psi \, d^2\mathbf{x} + \int_{-\infty}^{+\infty} \left[(\nabla\,\psi)^2 - 1 - 2\lambda\zeta\right] \delta\zeta \, dx \\ &= -2 \iint\limits_{\mathcal{D}} \Delta\psi \, \delta\psi \, d^2\mathbf{x} + 2 \oint\limits_{\partial\mathcal{D}} \delta\psi \psi_\nu \, ds + \int_{-\infty}^{+\infty} \left[(\nabla\,\psi)^2 - 1 - 2\lambda\zeta\right] \delta\zeta \, dx. \end{aligned}$$

The integral over the bottom of the flow domain vanishes, since $\psi_\nu = 0$ there. On the free surface (see $\delta(2)$, p. 65 [9])

$$\delta\psi + \psi_y \delta\zeta = 0.$$

This follows immediately by differentiating the relation $\psi_\varepsilon(x, 1+\zeta_\varepsilon(x)) \equiv 1$ with respect to ε and setting ε equal to zero. Similarly, differentiating the expression $\psi(x, 1+\zeta(x)) \equiv 1$ with respect to x we find that $\psi_x/\psi_y = -\zeta_x$; hence

$$\psi_\nu = \nabla\psi \cdot \nu = \nabla\psi \cdot \frac{\nabla\psi}{||\nabla\psi||} = \frac{(\nabla\psi)^2}{\sqrt{\psi_x^2+\psi_y^2}} = \frac{(\nabla\psi)^2}{|\psi_y|\sqrt{1+\zeta_x^2}}.$$

Hence δJ reduces to

$$\delta J = -2\iint_{\mathcal{D}} \Delta\psi\,\delta\psi\,d^2\mathbf{x} - \int_{-\infty}^{+\infty} \left[(\nabla\psi)^2 + 1 + 2\lambda\zeta\right]\delta\zeta\,dx. \tag{6.1}$$

First restrict the variations to fixed domains, $\delta\zeta = 0$, and the first integral must vanish for all variations $\delta\psi$ which vanish on $\partial\mathcal{D}$. Hence ψ is harmonic in the interior of $\mathcal{D}$, and the double integral vanishes.

We next consider variations of the domain. Since $\int \zeta_\varepsilon\,dx = m$, for all variations, we have $\int \delta\zeta\,dx = 0$; then the condition

$$\int_{-\infty}^{+\infty} ((\nabla\psi)^2 + 1 + 2\lambda\zeta)\delta\zeta\,dx = 0$$

for all such $\delta\zeta$ implies that the integrand is a constant. We therefore have $(\nabla\psi)^2 + 2\lambda\zeta + 1 = C = \mathit{const}.$ on the line; letting $x \to \infty$ and noting that $\zeta \to 0$ while $(\nabla\psi)^2 \to 1$ we see that $C = 2$, and the Bernoulli equation is satisfied. $\square$

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